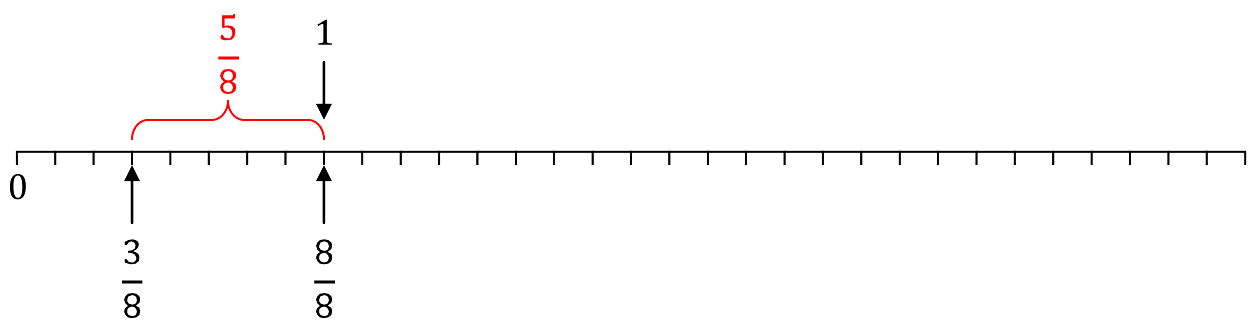
Multiplicative Inverse: Reciprocals

How does $\frac{3}{8}$ get to be 1? There is the old rhyme, “Ours is not to reason why: just invert and multiply.” But in these pages we’re going to take some time to discuss and picture the reasoning why.

Addition

The simplest way to get $\frac{3}{8}$ to be 1 is by addition: just add $\frac{5}{8}$ more to $\frac{3}{8}$ and you get 1:

On a number line, that looks something like this:

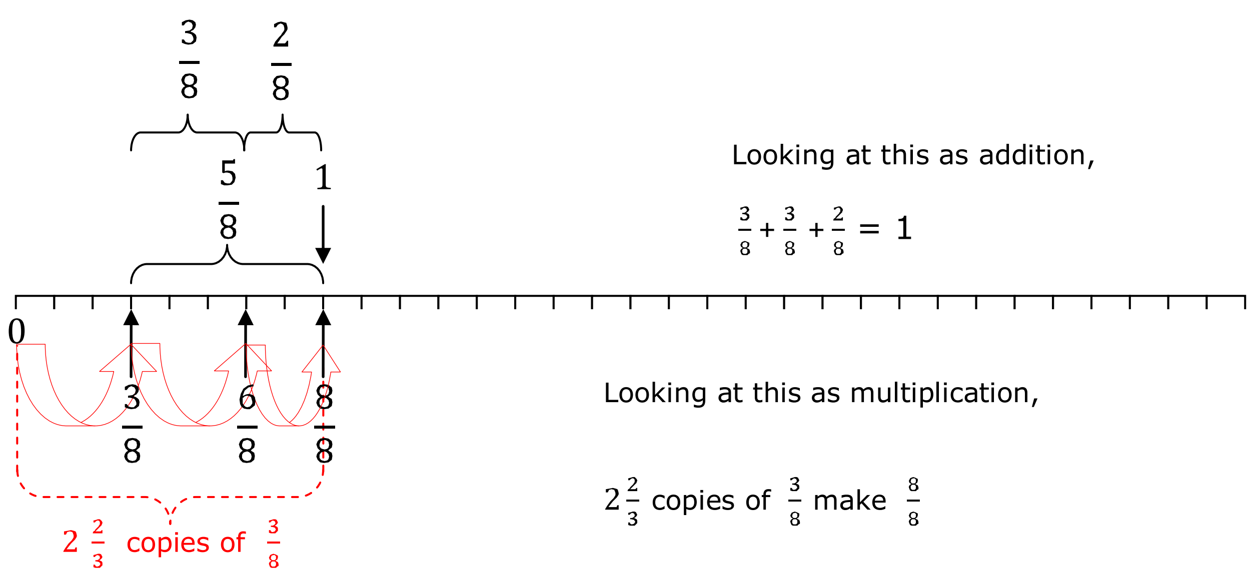


Moving from Addition to Multiplication

If we keep the $\frac{3}{8}$ in mind and think of things in terms of that length, then we’ll see the $\frac{5}{8}$ as $\frac{3}{8}$ plus $\frac{2}{8}$

So we have 3 copies of $\frac{1}{8}$ plus another 3 copies of $\frac{1}{8}$ and then 2 copies of $\frac{1}{8}$

(This should start feeling a lot like multiplication—or division.)



Looking at this as addition,

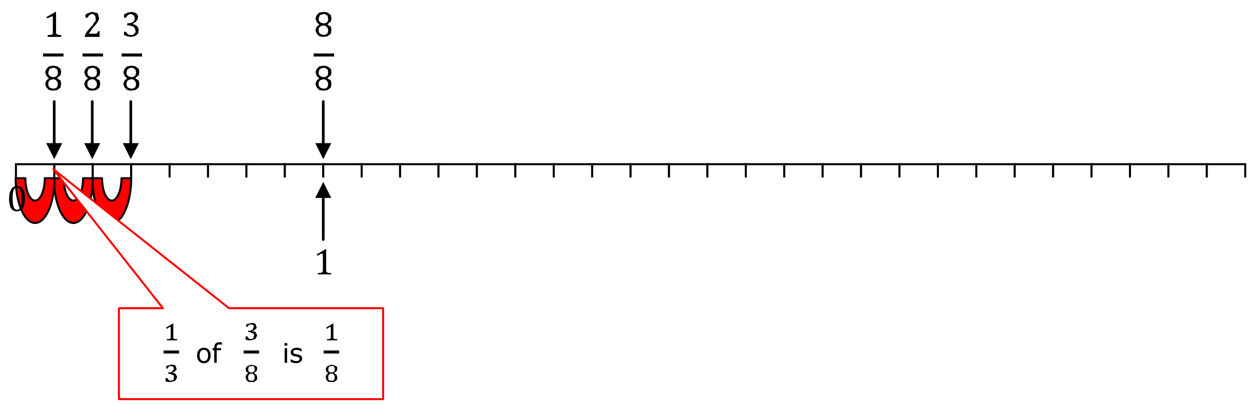
$\frac{3}{8} + \frac{3}{8} + \frac{2}{8} = 1$

Looking at this as multiplication,

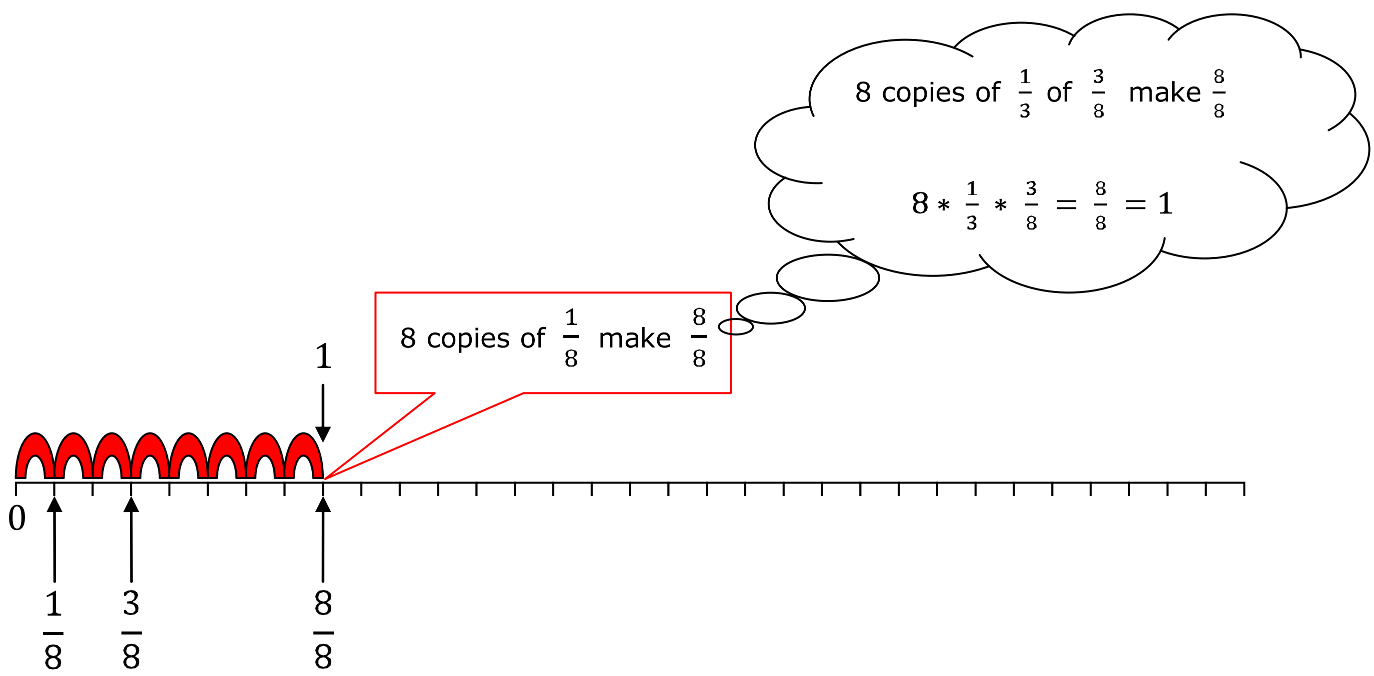
$2\frac{2}{3}$ copies of $\frac{3}{8}$ make $\frac{8}{8}$

Written symbolically $2\frac{2}{3} * \frac{3}{8} = 1$

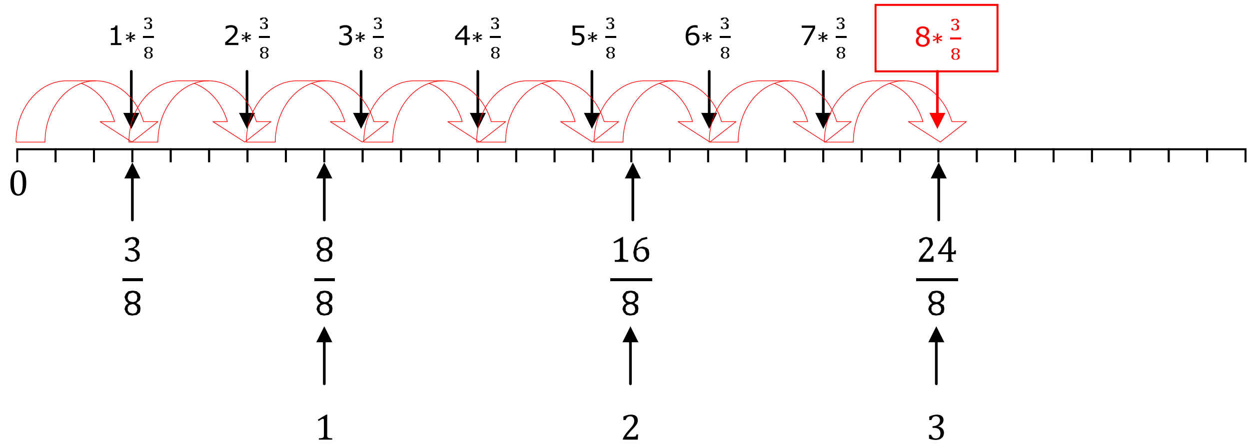
If we look again at the number line and recall our mission of transforming $\frac{3}{8}$ into 1, we could/should notice that we could divide $\frac{3}{8}$ into 3 equal pieces: then each one would equal $\frac{1}{8}$



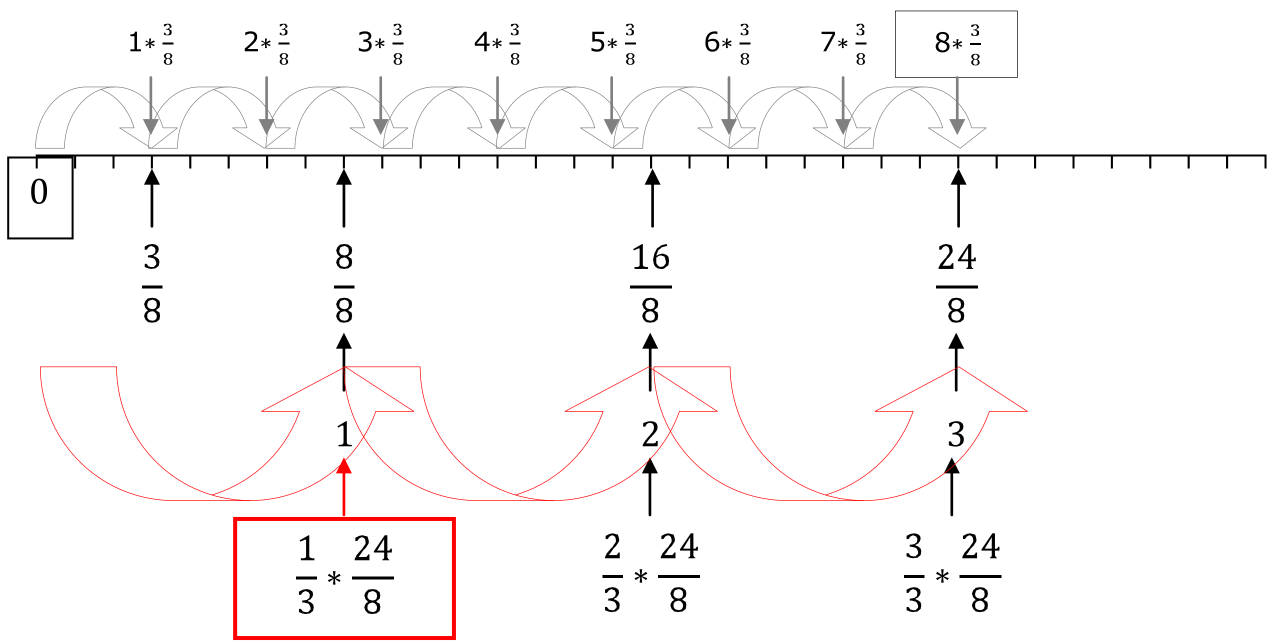
We could make 8 copies of that $\frac{1}{8}$ and we'd have $\frac{8}{8}$



One more way, of course, is to make 8 copies of $\frac{3}{8}$



And then take $\frac{1}{3}$ of that



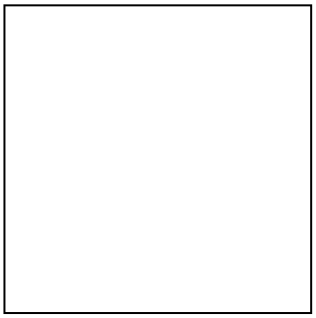
which will be 1, of course, since one-third of eight copies of three-eighths is one:

$$\frac{1}{3} * 8 * \frac{3}{8} = \frac{24}{24} = 1$$

Invoking the Commutative Property and simplifying, we get $\frac{3}{8} * \frac{8}{3} = 1$

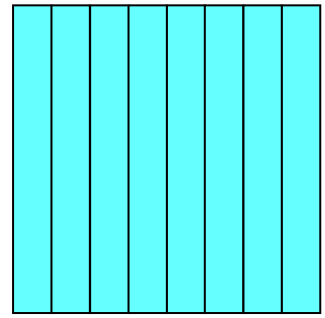
So multiplying $\frac{3}{8}$ by its reciprocal $\frac{8}{3}$ gives us 1. (It turns out that multiplying any number, $\frac{m}{n}$, for example, by its reciprocal, $\frac{n}{m}$, will give us 1, but we'll leave that demonstration for another paper.)

Instead of picturing this one-dimensionally, as length on a number line, we can picture this two-dimensionally, as area. Let’s think of 1 as a flat 1 x 1 unit square

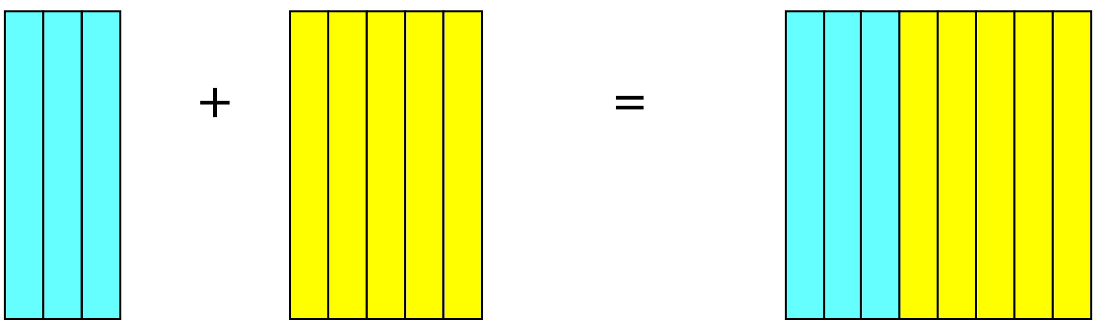


We can think of the unit square as 1 x $\frac{8}{8}$,
Dividing it horizontally into eighths

and picture it like this →

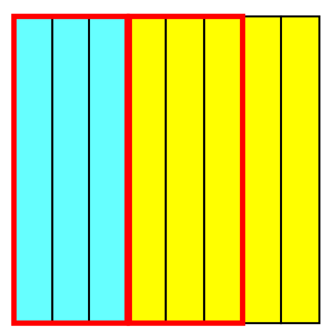


Then $\frac{3}{8} + \frac{5}{8} = 1$ will look something like this:



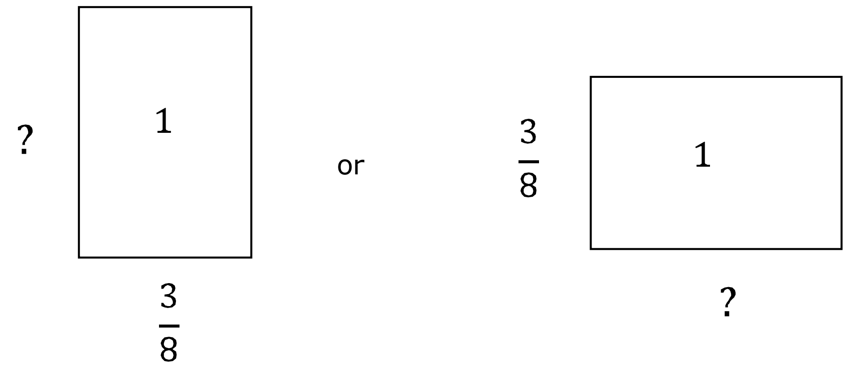
$\frac{3}{8}$ $\frac{5}{8}$ $\frac{8}{8}$

Since multiplication is an extension of addition, we ought to be able to re-arrange things a little and look at $\frac{3}{8}$ becoming 1 from a multiplication point of view. We can see the $\frac{3}{8}$ copies within the 1x1 unit square. (This should seem a lot like what we just did on the number line.)

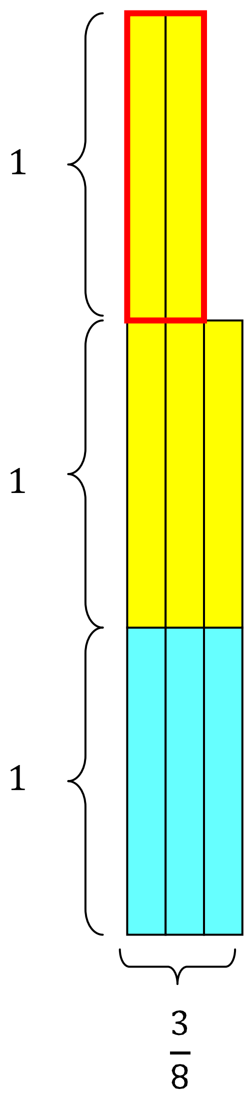


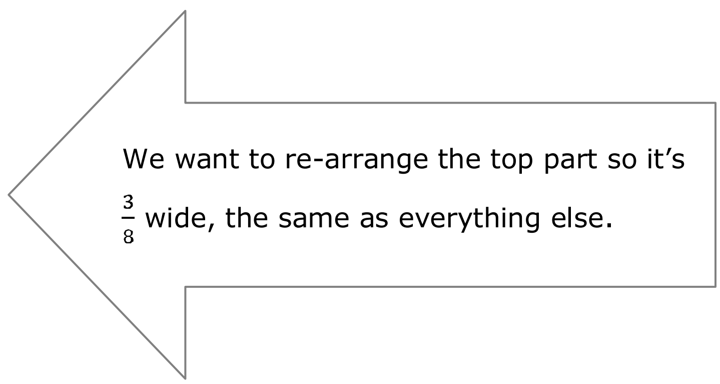
$\frac{3}{8}$ $\frac{3}{8}$ $\frac{2}{8}$

We know that $\frac{3}{8}$ times some number will equal 1. We can find out that number by re-arranging our 1 in terms of that $\frac{3}{8}$. Schematically, but not to scale, we have:



Let's get the $\frac{3}{8}$ + $\frac{5}{8}$ which we know adds up to 1 and arrange the $\frac{1}{8}$ pieces in a stack that's $\frac{3}{8}$ wide:

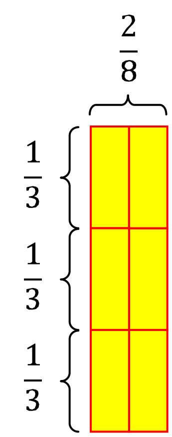


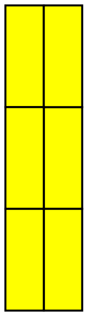
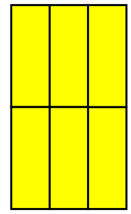


We divide each of those

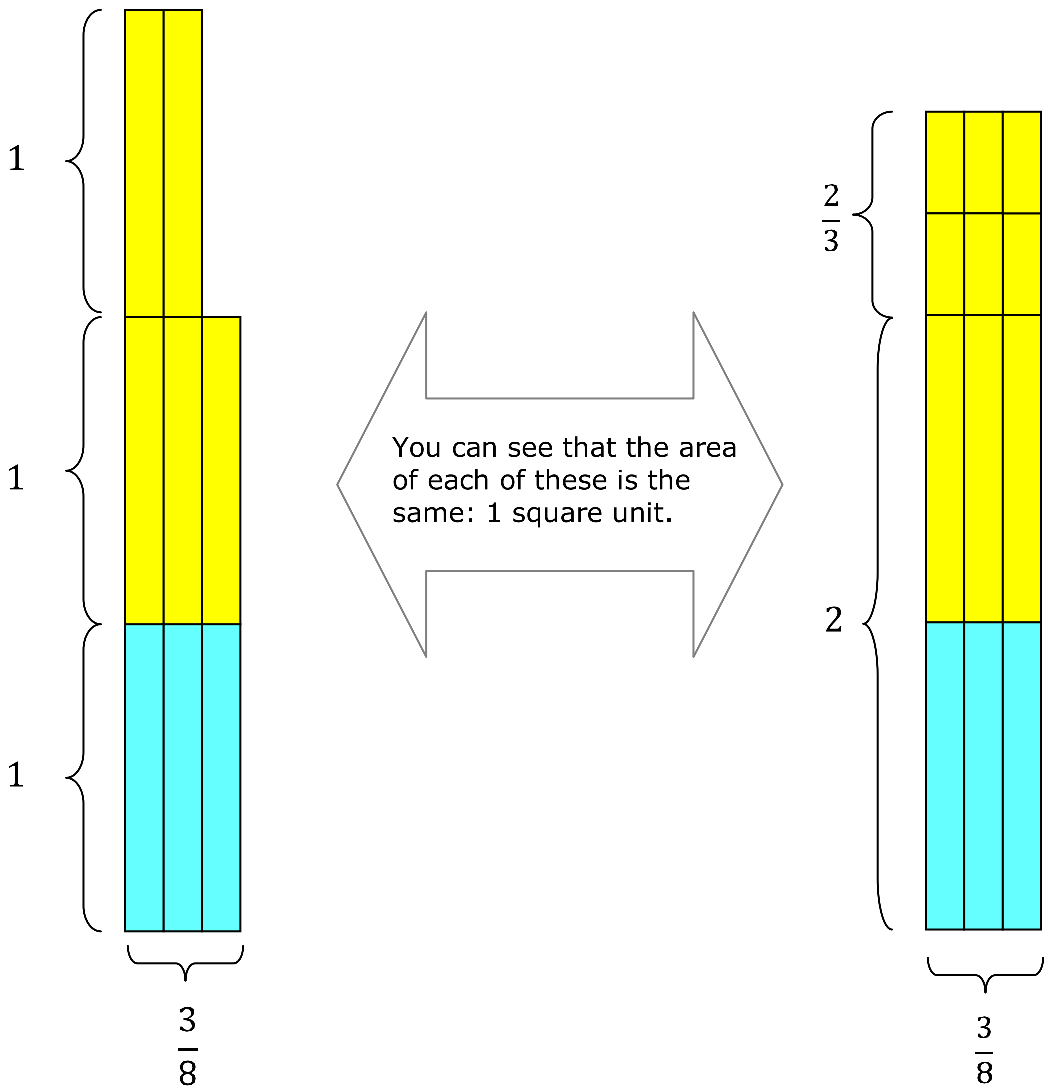
1 x $\frac{1}{8}$ strips into 3 equal parts

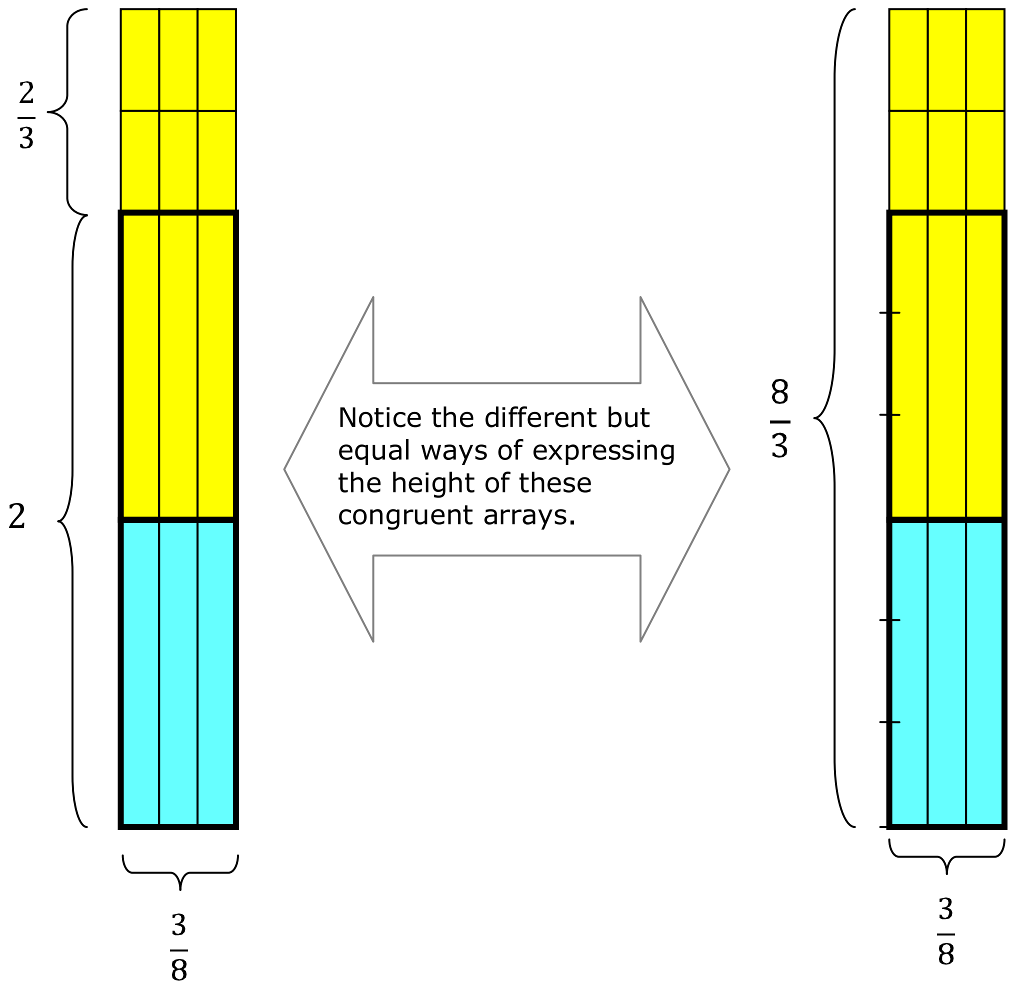
by dividing the height vertically into $\frac{3}{3}$



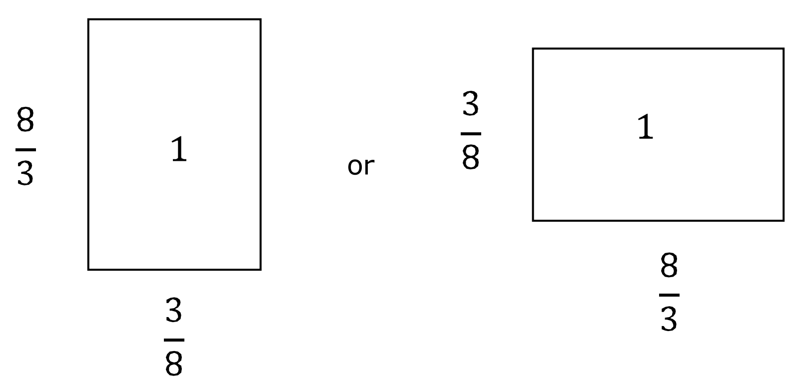
$\frac{3}{3}$ of $\frac{2}{8}$ looks like this  but we can re-arrange it like this: 

We can distribute those 6 pieces across the top of our stack and have a rectangle:





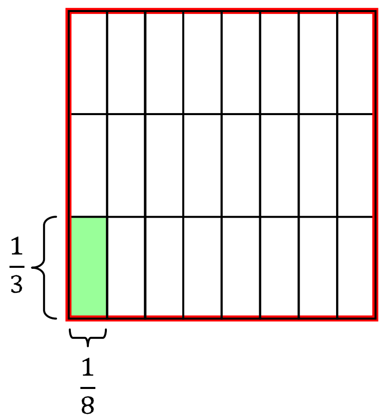
So, schematically, we have



And symbolically, we have, of course, $2\frac{2}{3} * \frac{3}{8} = 1$ or just $\frac{8}{3} * \frac{3}{8} = 1$

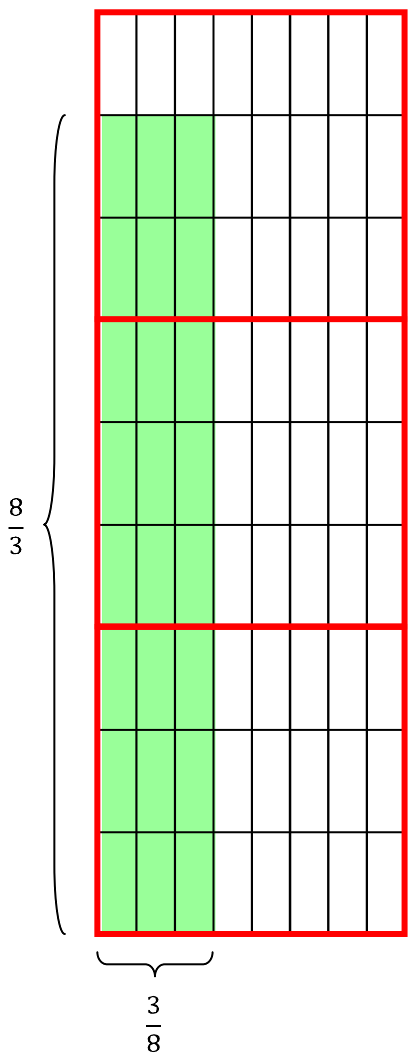
The invert-and-multiply algorithm that tells us we have to multiply $\frac{3}{8}$ by its reciprocal. When we multiply, we get $\frac{8}{3} * \frac{3}{8} = \frac{24}{24} = 1$. Let's take a look at our unit square to see where the 24ths come from.

If we divide the 1 unit height into thirds, we have a $\frac{3}{3}$ x $\frac{8}{8}$ unit square



Each little rectangle is $\frac{1}{24}$ of the unit square.

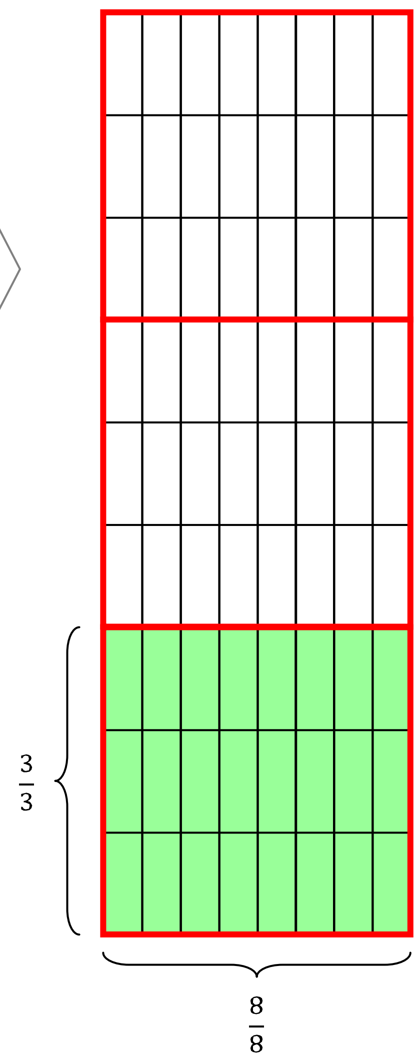
(You could count, or just recall that 8 x 3 = 24.)



You can see that both of these are equivalent. The area of each one of them is 1 square unit.

$$\frac{8}{3} * \frac{3}{8} = \frac{24}{24} = 1$$

$$\frac{3}{3} * \frac{8}{8} = \frac{24}{24} = 1$$



Conclusion

There are a few things that I'm hoping any reader would notice from the previous eight pages:

1. The number line model and the area model should remind you of each other in the way they look and the way they behave.

2. In the same way that **sums imply differences**, for example, once we say $\frac{3}{8} + \frac{5}{8} = 1$,

   that implies 1 - $\frac{3}{8} = \frac{5}{8}$ and 1 - $\frac{5}{8} = \frac{3}{8}$,

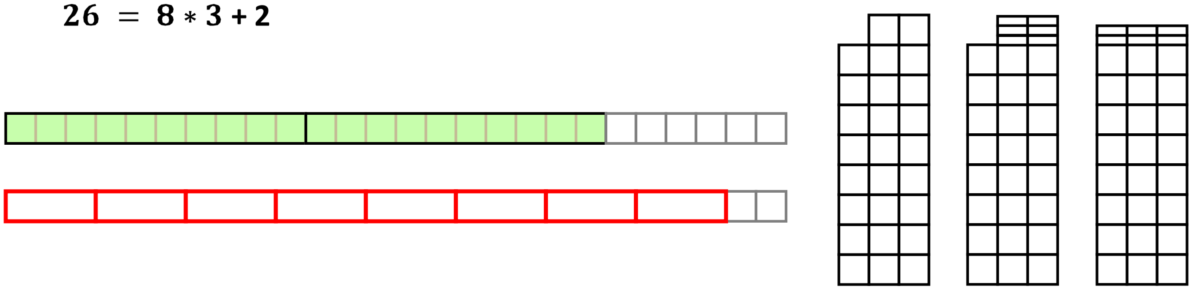
   products imply quotients: the product statement $2\frac{2}{3} * \frac{3}{8} = 1$

   implies a division statement: $\frac{1}{\frac{3}{8}} = \frac{8}{3}$ and also $\frac{1}{\frac{8}{3}} = \frac{3}{8}$

1	2	3
4	5	6
7	8	9
10	11	12
13	14	15
16	17	18
19	20	21
22	23	24
25	26	

3. The expression of the dividend (1, in this case) and stacking of $\frac{3}{8}$ pieces (the divisor) into rectangular arrays and then, if necessary, chopping up the leftovers, so they can be distributed across a full width, ought to remind everyone of **Division With Remainder**. Here's are some examples of Division With Remainder:

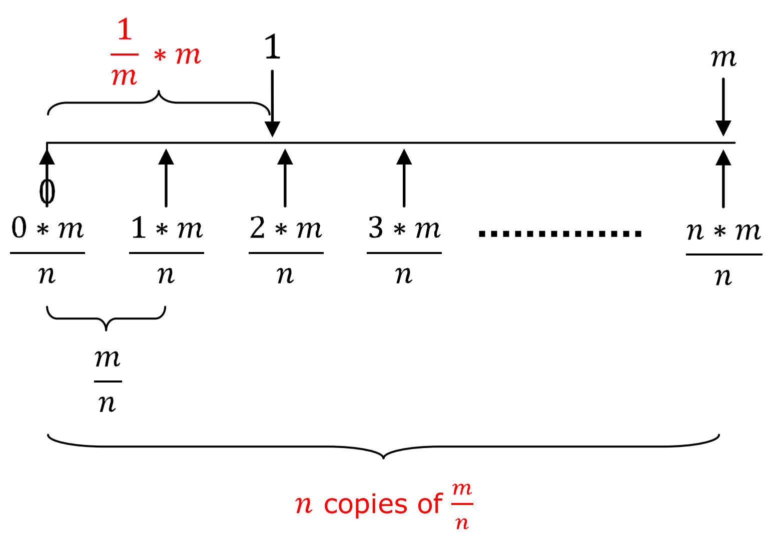
   $\mathbf{26 = 8 * 3 + 2}$



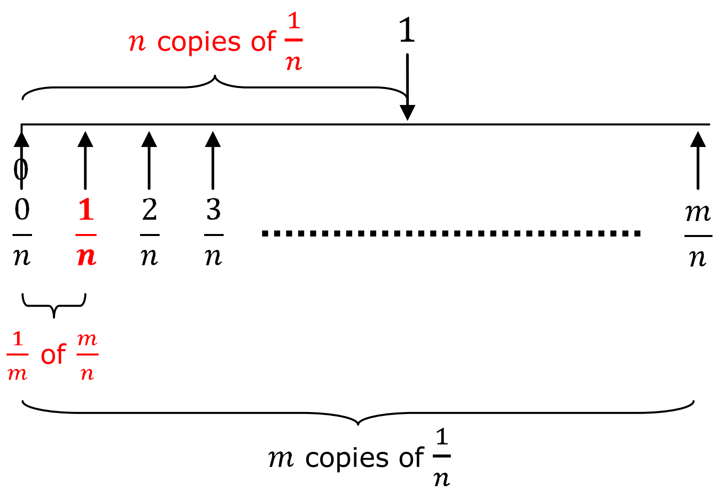
4. Counting underlies all this: the way to work successfully with fractions is to get them to be whole numbers or have common denominators so they can be **counted**: when we have $\frac{3}{8}$ we try to get to $\frac{1}{8}$ so we can make 8 copies and get to 1, or, alternatively, when we have $\frac{3}{8}$ we multiply by 8 to get to 3 so then we can take $\frac{1}{3}$ of that and get to 1. Generally, if we have a fraction, $\frac{m}{n}$ (where m and n are whole numbers). We locate it on the number line: that defines a segment $[\,0, \frac{m}{n}\,]$. If we want to multiply it by some number and have the product equal 1,

   a. we can make n copies of $\frac{m}{n}$ which will equal the segment

      $[\,0, m\,]$. We can divide that segment into m equal pieces. Each piece will be 1 unit long.



b. or we can divide our fraction $\frac{m}{n}$ by the numerator, m, and get a segment of length $\frac{1}{n}$. We can make n copies of that segment. The sum of all those copies will equal 1.



5. **Unit length and unit area** can be pictured on the same number line: the **segment [0, 1]** is the unit of length AND the **1x1 square** is the unit of area.

6. And **what's so important about reciprocals, anyway?** Here are two specific applications and a general one:

   a. When you're trying to solve an equation and you have a coefficient of x that's not 1, it's handy to know that if you multiply it by its reciprocal, you get 1.

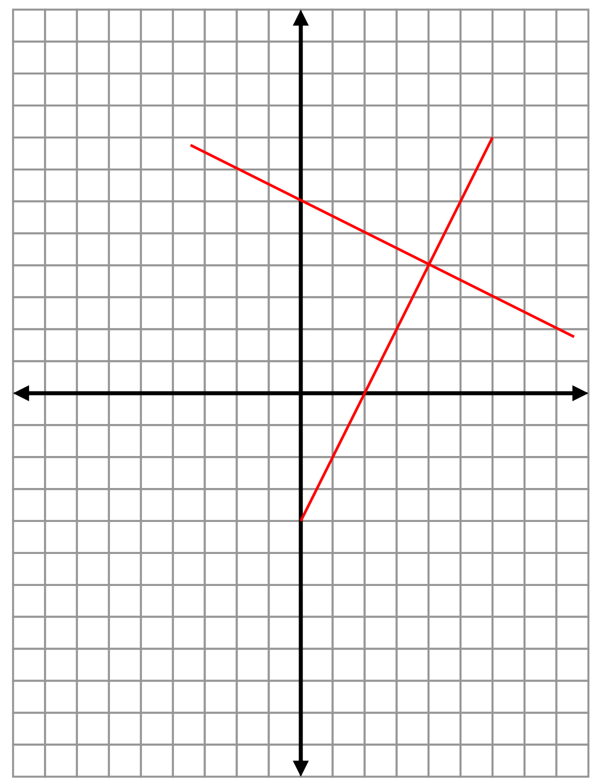
   $\frac{5}{4}x = 20$ *[given]*

   $\frac{4}{5} * \frac{5}{4}x = \frac{4}{5} * 20$ *[equivalent fractions]*

   $x = \frac{4}{5} * 5 * 4$ *[inverse property]*

   $x = \frac{5}{5} * 4 * 4$ *[commutative property]*

   $x = 16$ *[identity property]*



   b. When you want a line perpendicular to any given line, the slopes of the lines will be the negative reciprocals of each other.

   c. Reciprocals are where fractions, multiplication, division, and the commutative, identity, and inverse properties all meet.

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