Quadratic Formula—arrayed in all its glory

We are used to solving linear equations and finding out what x value(s) make them true. If we know that $4x + 4 = 7 + x$, by getting the x-terms on the left and the constants on the right (subtracting x from both sides and subtracting 3 from both sides and dividing both sides by 3), we find out that when x=1, the equation is true. We can solve ANY linear equation in this way.

In a quadratic equation, it's not as easy—unless we can factor the equation. For example, if we know that $2x^2 + 7x + 6 = 0$, and then we find that we can factor the equation into $(2x+3) \bullet (x+2) = 0$, we can solve for x. All we have to do is set each factor = 0 and find out that $x = -2$ and $x = -\frac{3}{2}$ both satisfy the equation, because zero times anything is zero.

But what about equations we can't factor? It turns out that by doing something called **completing the square**, we can get the left side of the equation in the form of a square. Then we can take its square root, isolate x, and solve for x. Let's look at it algebraically, using the letters a and b for the coefficients of x^2 and x and the letter c for the constant. This way we'll be able to come up with a formula, "The Quadratic Formula", that allows us to solve any quadratic equation.

$$ax^2 + bx + c = 0$$ standard form of any quadratic equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$ divide by a

$$x^2 + \frac{b}{a}x + \quad = -\frac{c}{a}$$ subtracting $\frac{c}{a}$ from both sides or adding $-\frac{c}{a}$ to both sides

Now things are in a form where we can solve for x by completing the square.

If you have an equation, you have

left side = *right side*

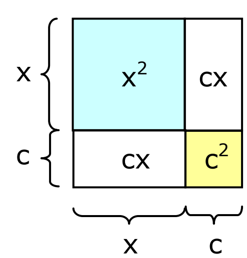
And, since it's an equality, if you do the same thing to both sides, they will still be equal.

So $\sqrt{\text{left side}} = \sqrt{\text{right side}}$

We need to add *something* to $x^2 + \frac{b}{a}x$ so that the left side will factor to $(x + \textit{something})^2$

(This way we can take the square root of both sides and solve the whole thing.)

What could that *something* be?



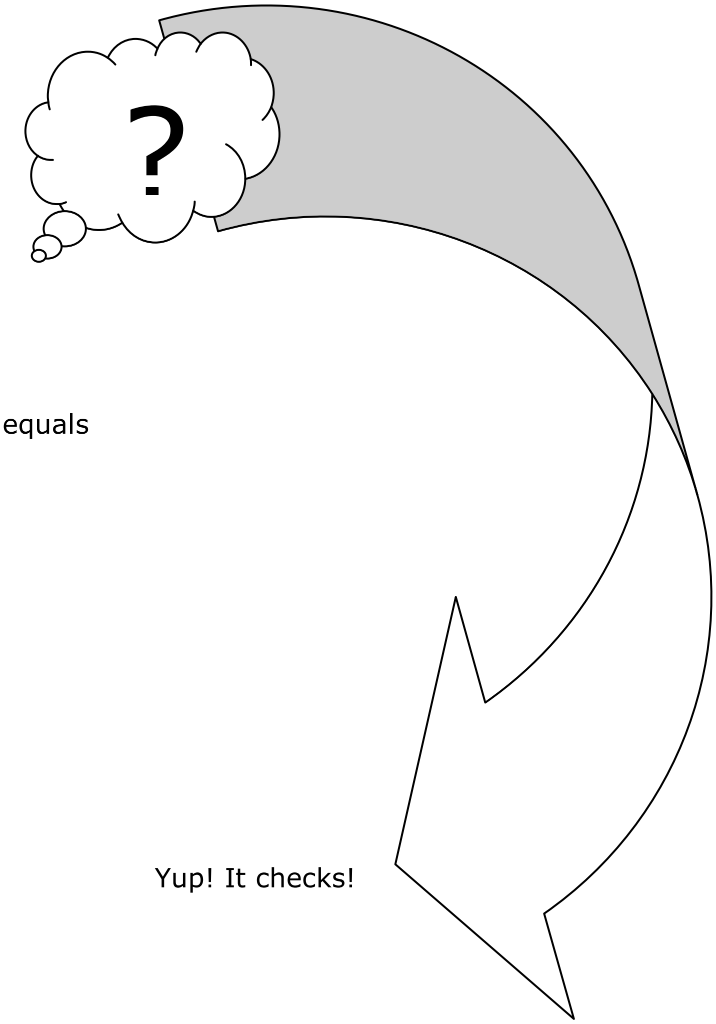
We know that $(x + c)^2$ expands to $x^2 + 2cx + c^2$

So our $\frac{b}{a} = 2c$ That's the $\frac{b}{a}$ coefficient of x in our original equation

So $\frac{b}{2a} = c$ dividing both sides by 2

Then c^2 will be $\left(\frac{b}{2a}\right)^2$ or $\frac{b^2}{4a^2}$

This is the *something* we have to add!



Let's check it:

Does $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$ equal $\left(x + \frac{b}{2a}\right)^2$

We'll multiply to find out what $\left(x + \frac{b}{2a}\right)^2$ equals

$$\begin{array}{rrrrr} & x & + & \frac{b}{2a} \\ & x & + & \frac{b}{2a} \\ \hline x^2 & + \frac{\cancel{2}b}{\cancel{2}a}x & + & \frac{b^2}{4a^2} \\ x^2 & + \frac{b}{a}x & + & \frac{b^2}{4a^2} \end{array}$$

Yup! It checks!

So that means we can add $\frac{b^2}{4a^2}$ to both sides to complete the square on the left side

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

Now we can express the left side of the equation as a quantity squared

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \bullet \frac{4a}{4a}$$

We get a common denominator or $4a^2$ by multiplying top and bottom by 4a. We can do this because of the identity property of multiplication:

$$4a \bullet 1 = 4a$$

$$\text{or } \frac{4a}{4a} = 1$$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

Now we can solve for x, since $\sqrt{\text{left side}} = \sqrt{\text{right side}}$

$$\sqrt{\left(x+\frac{b}{2a}\right)^2} = \sqrt{\frac{b^2-4ac}{4a^2}}$$

The square root of something squared is just something.

Getting the square root of $4a^2$

we have $\sqrt{4a^2} = \sqrt{2^2} \bullet \sqrt{a^2} = 2a$

$$x+\frac{b}{2a} = \frac{\sqrt{b^2-4ac}}{2a}$$

So $$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

Getting a common denominator…

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

This is it!

Now we have the quadratic formula:

> Given any quadratic equation in the form
>
> $ax^2 + bx + c = 0$
>
> We can solve for x, using the following equality:
>
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

> The form is important!
>
> Otherwise, when you use the formula to solve for x, the sign for c will be wrong—and your answer will be wrong, too!

Let's take a look at our quadratic formula applied to a known quadratic equation:

This one will be "known" because we'll start with the factors and work up to the equation itself.

$(2x + 3)(x + 1) = 0$

> We know the solutions to this one are
>
> $x = -\frac{3}{2}$ and $x = -1$
>
> because those make the factors
> (2x + 3) and (x + 1), respectively, equal zero.
> And zero times anything will equal zero.

Now we'll expand it and get the equation

$2x^2 + 3x + 2x + 3 = 0$ expanded

$2x^2 + 5x + 3 = 0$ simplified

So let's try $2x^2 + 5x + 3 = 0$

and solve for x using the quadratic formula—pretending we didn't already know the answers

Given $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

So $x = \frac{-5 \pm \sqrt{5^2 - 4(2)(3)}}{2(2)}$

$x = \frac{-5 \pm \sqrt{25 - 24}}{4}$

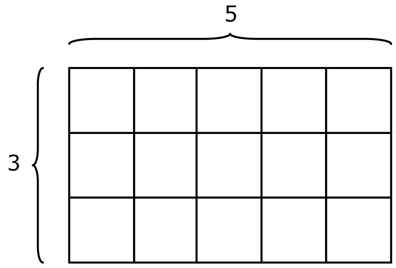
$x = \frac{-5 \pm 1}{4}$

> Since our equation is in standard form, the coefficients of x^2 and x in our equation,
>
> $2x^2 + 5x + 3 = 0$
>
> map onto a and b, respectively, and the constant maps onto c.
>
> So that in the quadratic formula,
> a = 2, b = 5, and c =3

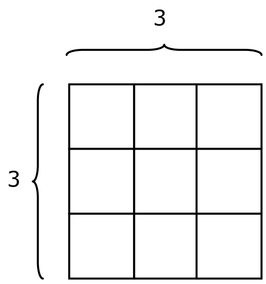
> $x = -1$
>
> and $x = -\frac{3}{2}$

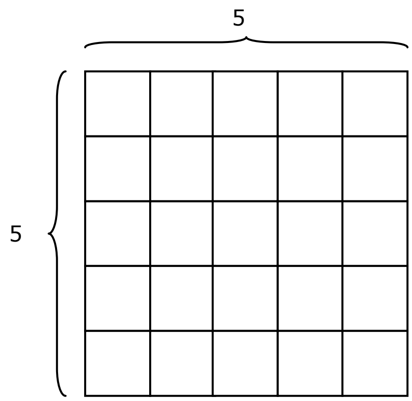
Let’s look at this another way—completing the square visually as well as symbolically.

The area of any rectangle is the product of its dimensions:

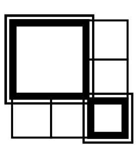


So the area of a square will be the length of any of its sides times itself (squared):

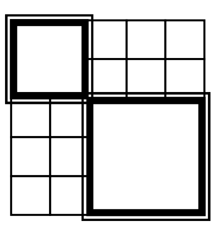




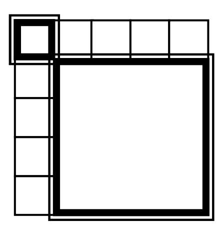
All squares, just like the 3x3 and 5x5 squares above, can be seen as made up of square pairs and partial products, just like our $(x + c)^2$ example from page 1:



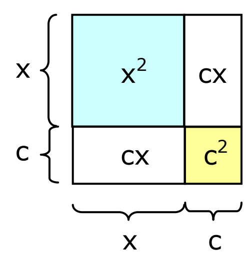
$2^2 + 1^2 + 2 \bullet 1 + 1 \bullet 2$



$2^2 + 3^2 + 2 \bullet 3 + 3 \bullet 2$

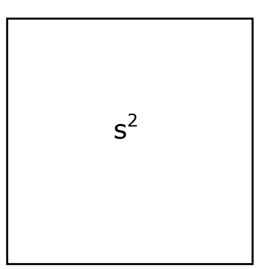


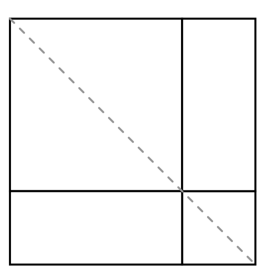
$1^2 + 4^2 + 1 \bullet 4 + 4 \bullet 1$



$x^2 + c^2 + x \bullet c + c \bullet x$

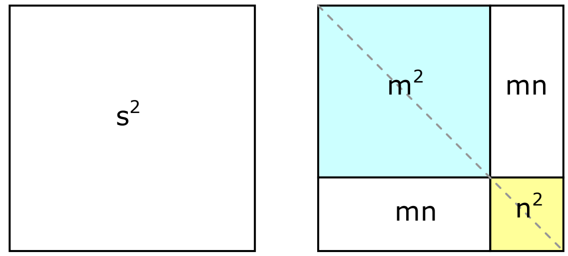
Actually, a square can be thought of as made up of (infinitely) many square-pairs and partial products whose corners meet along the diagonal of the big square:





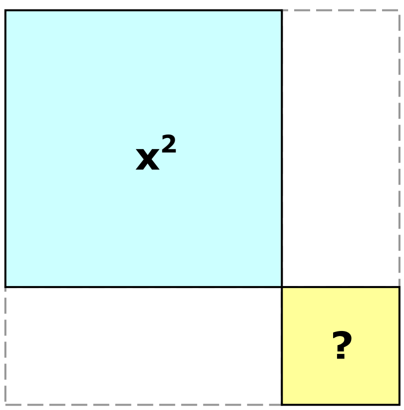
The big square (which we could call s^2) always equals the sum of the little squares (which we could call m^2 and n^2, plus twice the products (rectangles) of their sides.

Put symbolically, $s^2 = (m + n)^2 = m^2 + 2mn + n^2$



Now let's get back to our original complete-the-square case…

We had $\mathbf{x^2 + \frac{b}{a}x + ?}$



Completing the square is like having a square-pair where one of the squares we know and the other we have to figure out.

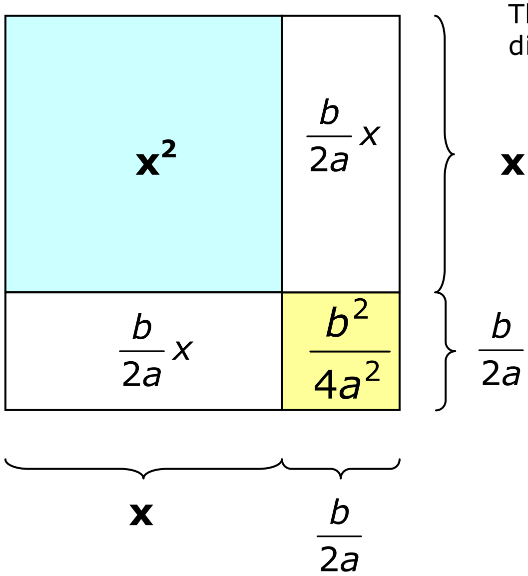
In this case, one of the squares in the square-pair is x^2. But what's the other one?

We know that **the partial products will add up to** $\frac{b}{a}\mathbf{x}$ and we know they are equal—since it's a square.

Their being equal means that each one is $\frac{1}{2}$ of the total,

so each one has to be $\frac{1}{2} \bullet \frac{b}{a}$ which simplifies to $\frac{b}{2a}$

This tells us what we need to know to fill in the dimensions of our square.



The dimensions tell us we can replace the **?**

with $\frac{b}{2a} \bullet \frac{b}{2a}$ which simplifies to $\frac{b^2}{4a^2}$

We have completed the square!

References:

en.wikipedia.org/wiki/Completing_the_square and en.wikipedia.org/wiki/Quadratic_formula
planetmath.org/encyclopedia/DerivationOfQuadraticFormula.html and planetmath.org/encyclopedia/CompletingTheSquare.html

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